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SOME PROPERTIES OF QUASI CLASS Q OPERATORS

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ABSTRACT

An operator T on a complex Hilbert space is said to be Quasi – Class Q if $T^{*3}T^3 - 2T^{*2}T^2 + T^*T \geq 0$. In this paper the characterization and some basic algebraic properties of Quasi –Class Q operators are investigated.

MATHEMATICS SUBJECT CLASSIFICATION: 47B20, 47B99, 47B15

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INTRODUCTION

In this paper it is introduced and studied a new class of operators called Quasi –Class Q operators. Let H be a finite dimensional complex (Separable) Hilbert space. Let T be a bounded linear operator defined on H. During the last decades, several classes of bounded linear operators are studied intensively. The classes of normal, Quasinormal, hyponormal, Quasihyponormal and paranormal operators are well known. Many properties of these classes are investigated by many authors like, S.M.Patel, Mahmoud M.Kutkut and others. The following proper inclusion relation holds for these classes .

Normal $\subset$ Quasinormal $\subset$ Subnormal $\subset$ Hyponormal

$\subset$ Quasihyponormal $\subset$ Paranormal.

An operator $T \in B(H)$ is said to be hyponormal [3], if $TT^* \leq T^*T$, or equivalently $\|T^*x\| \leq \|Tx\|$, for all $x \in H$; and Quasihyponormal [3], if $\|T^*T\| \leq \|T^2 x\|$, for all $x \in H$ or equivalently $T^{*2}T^2 - (T^*T)^2 \leq 0$. An operator $T \in B(H)$ is called paranormal [3], if $\|Tx\|^2 \leq \|T^2 x\|$, for all $x \in H$; with $\|x\| = 1$. An operator T in B(H) is said to be quasiparanormal if $\|T^*T x\|^2 \leq \|T^3 x\| \, \|T x\|$, for all $x \in H$; with $\|x\|=1$.Motivated by these we introduce the following operator. In [2] B.P.Duggal, C.S.Kubrusly and N.Levan introduced the class Q operator as follows.

Definition: 1[2]

An operator T on a Hilbert space H is said to be Class Q

if $T^{*2}T^2 - 2T^*T + I \geq 0$.

Definition: 2

An operator T on a Hilbert space H is said to be Quasi – Class Q

if $T^{*3}T^3 - 2T^{*2}T^2 + T^*T \geq 0$.

In this paper, some properties of Quasi – Class Q operators are studied. The characterization of the Quasi – Class Q operators and some results on invertible operator, unitary, Aluthge transformation and other basic properties of Quasi – Class Q operators are investigated.

Theorem: 1.1

Let $T\in B(H)$. Then T is Quasi – Class Q if and only if

$\|T^2 x\|^2 \leq 1/2\ (\|T^3 x\|^2 + \|T x\|^2)$

Proof

Assume that T is Quasi – Class Q operator.

$\Leftrightarrow T^{*3}T^3 - 2T^{*2}T^2 + T^*T \geq 0$

$\Leftrightarrow \langle (T^{*3}T^3 - 2T^{*2}T^2 + T^*T)\, x, x \rangle \geq 0$, for any $x\in H$

$\Leftrightarrow \langle T^3 x, T^3 x\rangle - \langle 2T^2 x, T^2 x\rangle + \langle Tx, Tx\rangle \geq 0$, for any $x\in H$

$\Leftrightarrow \|T^3x\|^2 - 2\|T^2x\|^2 + \|Tx\|^2 \geq 0$, for any $x\in H$

$\Leftrightarrow \|T^2 x\|^2 \leq 1/2\ (\|T^3 x\|^2 + \|T x\|^2)$, for any $x\in H$

Theorem: 1.2

Let T be an invertible operator on H and N be an operator such that N commutes with T^*T. Then N is Quasi – Class Q if and only if TNT^{-1} is of Quasi – class Q.

Proof

Let N be a Quasi – Class Q operator.

Then $N^{*3}N^3 - 2N^{*2}N^2 + N^*N \geq 0$

Consider,

$(TNT^{-1})^{*3}(TNT^{-1})^3$

$= (TNT^{-1})^* (TNT^{-1})^* (TNT^{-1})^* (TNT^{-1}) (TNT^{-1}) (TNT^{-1})$

Since N Commutes with T^*T

On Simplifying, we get,

$= TN^{*3}N^3T^{-1}$………………. (1)

Similarly,

$(TNT^{-1})^{*2}(TNT^{-1})^2$

$=(TNT^{-1})^* (TNT^{-1})^* (TNT^{-1}) (TNT^{-1})$

$= TN^{*2}N^2T^{-1}$……………… (2)

And $(TNT^{-1})^* (TNT^{-1}) = TN^*NT^{-1}$ ………………. (3)

Therefore Consider,

$(TNT^{-1})^{*3} (TNT^{-1})^3 - 2(TNT^{-1})^{*2} (TNT^{-1})^2 + (TNT^{-1})^* (TNT^{-1})$

Substituting the equation (1),(2) and (3) in above expression we have

$TN^{*3}N^3T^{-1} - 2TN^{*2}N^2T^{-1} + TN^*NT^{-1}$

$= T\,(N^{*3}N^3 - 2N^{*2}N^2 + N^*N)\,T^{-1}$

Now ,

$\Rightarrow T\,(N^{*3}N^3 - 2N^{*2}N^2 + N^*N)\,T^* \geq 0$

Consider, $T\,(N^{*3}N^3 - 2N^{*2}N^2 + N^*N)\,T^*\,[TT^*]$

$= T\,(N^{*3}N^3 - 2N^{*2}N^2 + N^*N)\,[T^*\,T]T^*$

$= T[T^*T]\,(N^{*3}N^3 - 2N^{*2}N^2 + N^*N)\,T^*$

$= [TT^*]\,T\,(N^{*3}N^3 - 2N^{*2}N^2 + N^*N)\,T^*$

So that TT^* Commutes with $T\,(N^{*3}N^3 - 2N^{*2}N^2 + N^*N)\,T^*$

Then $[TT^*]^{-1}$ also commutes with

$T\,(N^{*3}N^3 - 2N^{*2}N^2 + N^*N)\,T^*$

But we know that " if A and B are positive operators and AB = BA then AB is also positive operator ''. Applying this results for operator $[TT^*]^{-1}$ and

$T\,(N^{*3}N^3 - 2N^{*2}N^2 + N^*N)\,T^*$

$\Rightarrow [T\,(N^{*3}N^3 - 2N^{*2}N^2 + N^*N)\,T^*][TT^*]^{-1} \geq 0$

$\Rightarrow T\,(N^{*3}N^3 - 2N^{*2}N^2 + N^*N)\,T^*T^{*-1}\,T^{-1} \geq 0$

$\Rightarrow T\,(N^{*3}N^3 - 2N^{*2}N^2 + N^*N)\,T^{-1} \geq 0$

$\Rightarrow T(N^{*3}N^3)\,T^{-1} - 2T(N^{*2}N^2\,)T^{-1} + T(N^*N)T^{-1} \geq 0$

TNT^{-1} is Quasi – Class Q.

Conversely,

Let TNT^{-1} is Quasi – Class Q

$\Rightarrow (TNT^{-1})^{*3}(TNT^{-1})^3 - 2(TNT^{-1})^{*2}(TNT^{-1})^2 + (TNT^{-1})^*\,(TNT^{-1}) \geq 0$

$\Rightarrow T\,(N^{*3}N^3 - 2N^{*2}N^2 + N^*N)T^{-1} \geq 0$

$\Rightarrow T^*T\,(N^{*3}N^3 - 2N^{*2}N^2 + N^*N)T^{-1}T \geq 0$

$T^*T\,(N^{*3}N^3 - 2N^{*2}N^2 + N^*N) \geq 0$

Since $[T^*T]$ Commutes with N and hence with

$[T^*T]\,(N^{*3}N^3 - 2N^{*2}N^2 + N^*N)$

Therefore $[T^*T]^{-1}$ Commutes with $[T^*T]\,(N^{*3}N^3 - 2N^{*2}N^2 + N^*N)$

Applying the result "If $T \geq 0$ and $S \geq 0$ if then $ST \geq 0$ if and only if $ST = TS$"

for the operator $[T^*T]^{-1}$ and $[T^*T]\,(N^{*3}N^3 - 2N^{*2}N^2 + N^*N)$ we have

$[T^*T]^{-1}\,[T^*T]\,(N^{*3}N^3 - 2N^{*2}N^2 + N^*N) \geq 0$

Therefore, $N^{*3}N^3 - 2N^{*2}N^2 + N^*N \geq 0$

Hence N is Quasi – Class Q.

Corollary: 1.3

Let S be a Quasi – Class Q operator and X any positive operator such that $X^{-1} = X^*$. Then $T = X^{-1}SX$ is Quasi – Class Q.

Theorem: 1.4

Let $T \in B(H)$ be an operator of Quasi – Class Q.

- The restriction of T to an invariant subspace is again a Quasi – Class Q operator.
- If T is invertible, then T^{-1} is of Quasi – Class Q operator.

Proof

Let T be an operator of Quasi – Class Q and Let M be a T - invariant subspace.

- If $u \in H$, then

$|T^3u||^2 - 2||T^2u||^2 + ||Tu||^2 \geq 0$

$\Rightarrow 2||T^2u||^2 \leq ||T^3u||^2 + ||Tu||^2$

if $u \in M$, then

$2||(T|_M)^2u||^2 = 2||T^2u||^2 \leq ||T^3u||^2 + ||Tu||^2 = ||(T|_M)^3u||^2 + ||T|_Mu||^2$

$2||(T|_M)^2u||^2 \leq ||(T|_M)^3u||^2 + ||T|_Mu||^2$

So $T|_M$ is Quasi – Class Q.

- If T is invertible, then

To prove $2||T^{-2}y||^2 \leq ||T^{-3}y||^2 + ||T^{-1}y||^2$

Consider,

$2||T^{-1}x||^2 = 2||T^{-1}T^{-1}(Tx)\ ||^2 \leq ||T^{-3}(Tx)\ ||^2 + ||T^{-1}(Tx)\ ||^2$ for every $x \in H$.

Take any $y \in H$ = ran (T) So that $y = Tx$, $x = T^{-1}y$ and $T^{-1}x = T^{-2}y$

for some $x \in H$.

$2||T^{-2}y||^2 \leq ||T^{-3}y||^2 + ||T^{-1}y||^2$

Hence T^{-1} is Quasi – Class Q

Definition: 3[4]

If $T \in B(H)$, Then the unitary orbit of T denoted by $U(T) = U^*TU$: U is unitary on H

Theorem: 1.5

Let $T \in B(H)$ and T is Quasi – Class Q, then the unitary orbit of T is

Quasi – Class Q.

Proof

T is Quasi – Class Q . Then

$T^{*3}T^3 - 2T^{*2}T^2 + T^*T \geq 0$ …………….. (1)

Consider,

$(U^*TU)^{*3}(U^*TU)^3 - 2(U^*TU)^{*2}(U^*TU)^2 + (U^*TU)^*(U^*TU) \geq 0$

Simplifying, we get

$U^*T^{*3}T^3U - 2(U^*T^{*2}T^2U) + U^*T^*TU \geq 0$

$\Rightarrow U^*(T^{*3}T^3 - 2T^{*2}T^2 + T^*T)U \geq 0$ by (1)

Hence U^*TU is Quasi – Class Q that is Unitary orbit of T is Quasi – Class Q.

Corollary: 1.6

If Quasi – Class Q operator T Commutes with an isometric operator S, then TS is Quasi – Class Q.

Definition: 4[5]

Let T be an operator whose polar decomposition is $T = U|T|$,

where $|T| = (T^*T)^{1/2}$. The operator $\tilde{T} = |T|^{1/2}U|T|^{1/2}$ is called Aluthge transformation.

Definition: 5[5]

Let $T = U|T|$ be the polar decomposition of an operator T. Then * - Aluthge transformation is defined by $\tilde{T}^{(*)} = |T^*|^{1/2}U|T^*|^{1/2}$.

Theorem: 1.7

Let $T \in B(H)$, then $\tilde{T}$ is Quasi – Class Q if and only if $\tilde{T}^{(*)}$ is Quasi – Class Q.

Proof

Assume that $\tilde{T}$ be Quasi – Class Q

$$\tilde{T}^{*3}\tilde{T}^3 - 2\tilde{T}^{*2}\tilde{T}^2 + \tilde{T}^*\tilde{T} \geq 0$$

To Prove: $\tilde{T}^{(*)}$ is Quasi – Class Q.

Now,

$(\tilde{T}^{(*)})^{*3}(\tilde{T}^{(*)})^3 - 2(\tilde{T}^{(*)})^{*2}(\tilde{T}^{(*)})^2 + (\tilde{T}^{(*)})^*\tilde{T}^{(*)}$

$= (|T^*|^{1/2}U|T^*|^{1/2})^{*3}(|T^*|^{1/2}U|T^*|^{1/2})^3 - 2(|T^*|^{1/2}U|T^*|^{1/2})^{*2}(|T^*|^{1/2}U|T^*|^{1/2})^2$

$+ (|T^*|^{1/2}U|T^*|^{1/2})^*(|T^*|^{1/2}U|T^*|^{1/2})$

$= (|T^*|^{1/2}U^*|T^*|^{1/2})(|T^*|^{1/2}U^*|T^*|^{1/2})(|T^*|^{1/2}U^*|T^*|^{1/2})(|T^*|^{1/2}U|T^*|^{1/2})$

$(|T^*|^{1/2}U|T^*|^{1/2})(|T^*|^{1/2}U|T^*|^{1/2}) - 2(|T^*|^{1/2}U^*|T^*|^{1/2})(|T^*|^{1/2}U^*|T^*|^{1/2})$

$(|T^*|^{1/2}U|T^*|^{1/2})(|T^*|^{1/2}U|T^*|^{1/2}) + (|T^*|^{1/2}U^*|T^*|^{1/2})(|T^*|^{1/2}U|T^*|^{1/2})$

$= U(|T^*|^{1/2}U^*|T^*|\,U^*|T^*|\,U^*|T^*|\,U|T^*|\,U|T^*|\,U|T^*|^{1/2} - 2|T^*|^{1/2}U^*|T^*|\,U^*|T^*|\,U|T^*|$

$U|T^*|^{1/2} + |T^*|^{1/2}U^*|T^*|\, U|T^*|^{1/2})U^*$

$= U|T|^{1/2}\, U^*|T|\, U^*|T|\, U^*|T|\, U|T|\, U|T|\, U|T|^{1/2}U^* - 2U|T|^{1/2}\, U^*|T|\, U^*|T|\, U|T|$

$U|T|^{1/2}U^* + U\, |T|^{1/2}U^*|T|\, U|T|^{1/2}\, U^*$

$= U\, (|T|^{1/2}\, U^*\, |T|^{1/2})\, (|T|^{1/2}\, U^*\, |T|^{1/2})\, (|T|^{1/2}\, U^*\, |T|^{1/2})\, (|T|^{1/2}\, U\, |T|^{1/2})$

$(|T|^{1/2}\, U\, |T|^{1/2})\, (|T|^{1/2}\, U\, |T|^{1/2})\, U^* – 2\, U(|T|^{1/2}\, U^*\, |T|^{1/2})\, (|T|^{1/2}\, U^*\, |T|^{1/2})$

$(|T|^{1/2}\, U|T|^{1/2})\, (|T|^{1/2}\, U|T|^{1/2})\, U^* + U\, (|T|^{1/2}\, U^*\, |T|^{1/2})\, (|T|^{1/2}\, U\, |T|^{1/2})\, U^*$

$U(\tilde{T}^{*\mathbf{3}}\tilde{T}^{\mathbf{3}} - \mathbf{2}\tilde{T}^{*\mathbf{2}}\tilde{T}^{\mathbf{2}} + \tilde{T}^*\tilde{T})U^* \geq 0$

Hence $\tilde{T}^{(*)}$ is Quasi – Class Q.

Conversely,

Assume $\tilde{T}^{(*)}$ is Quasi – Class Q.

To Prove $\tilde{T}$ is Quasi- Class Q.

ie: To Prove $\tilde{T}^{*\mathbf{3}}\tilde{T}^{\mathbf{3}} - \mathbf{2}\tilde{T}^{*\mathbf{2}}\tilde{T}^{\mathbf{2}} + \tilde{T}^*\tilde{T} \geq 0$

Consider,

$\tilde{T}^{*\mathbf{3}}\tilde{T}^{\mathbf{3}} - \mathbf{2}\tilde{T}^{*\mathbf{2}}\tilde{T}^{\mathbf{2}} + \tilde{T}^*\tilde{T}$

$= (|T|^{1/2}\, U\, |T|^{1/2})^{*3}(|T|^{1/2}\, U\, |T|^{1/2})^3 - 2\, (|T|^{1/2}\, U\, |T|^{1/2})^{*2}(|T|^{1/2}\, U\, |T|^{1/2})^2 + (|T|^{1/2}\, U\, |T|^{1/2})^*\, (|T|^{1/2}\, U\, |T|^{1/2})$

$= U\, (|T|^{1/2}\, U^*\, |T|^{1/2})\, (|T|^{1/2}\, U^*\, |T|^{1/2})\, (|T|^{1/2}\, U^*\, |T|^{1/2})\, (|T|^{1/2}\, U\, |T|^{1/2})\, (|T|^{1/2}\, U\, |T|^{1/2})\, (|T|^{1/2}\, U\, |T|^{1/2})\, U^* – 2\, U(|T|^{1/2}\, U^*$ $|T|^{1/2})\, (|T|^{1/2}\, U^*\, |T|^{1/2})\, (|T|^{1/2}\, U|T|^{1/2})\, (|T|^{1/2}\, U|T|^{1/2})\, U^* + U\, (|T|^{1/2}\, U^*\, |T|^{1/2})\, (|T|^{1/2}\, U\, |T|^{1/2})\, U^*$

$= U|T|^{1/2}\, U^*|T|\, U^*|T|\, U^*|T|\, U|T|\, U|T|\, U|T|^{1/2}U^* - 2U|T|^{1/2}\, U^*|T|\, U^*|T|\, U|T|\;\; U|T|^{1/2}U^* + U\, |T|^{1/2}U^*|T|\, U|T|^{1/2}\, U^*$

$= U(|T^*|^{1/2}\, U^*|T^*|\, U^*|T^*|\, U^*|T^*|\, U|T^*|\, U|T^*|\, U|T^*|^{1/2} - 2|T^*|^{1/2}\, U^*|T^*|\, U^*|T^*|\, U|T^*|\;\; U|T^*|^{1/2} + |T^*|^{1/2}U^*|T^*|\, U|T^*|^{1/2})U^*$

$= (|T^*|^{1/2}\, U\, |T^*|^{1/2})^{*3}(|T^*|^{1/2}\, U\, |T^*|^{1/2})^3 - 2\, (|T^*|^{1/2}\, U\, |T^*|^{1/2})^{*2}\, (|T^*|^{1/2}\, U\, |T^*|^{1/2})^2 + (|T^*|^{1/2}\, U\, |T^*|^{1/2})^*\, (|T^*|^{1/2}\, U\, |T^*|^{1/2})$

Since, $(\tilde{T}^{(*)})^{*\mathbf{3}}\, (\tilde{T}^{(*)})^{\mathbf{3}} - \mathbf{2}\, (\tilde{T}^{(*)})^{*\mathbf{2}}\, (\tilde{T}^{(*)})^{\mathbf{2}} + (\tilde{T}^{(*)})^{*}\, \tilde{T}^{(*)} \geq 0$

Therefore, $\tilde{T}^{*\mathbf{3}}\tilde{T}^{\mathbf{3}} - \mathbf{2}\tilde{T}^{*\mathbf{2}}\tilde{T}^{\mathbf{2}} + \tilde{T}^*\tilde{T} \geq 0$

Hence $\tilde{T}$ is Quasi – Class Q.

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